Charged particle drift in the Earth magnetic field

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Abstract

I have simulated the motion of charged particles on an magnetic bottle and in the Earth magnetic field. In both cases the particles bounce between the poles with a helix-like trajectory.

1. Introduction

Solar winds from the sun contains a lot of electrical charged particles which are deflected by the Earth magnetic field. The particles will experience the Lorenz force and move accordingly. This will result in the particles undergoing a 'bouncing' motion from pole to pole. A magnetic bottle made by two facing Helmholtz coils will set up a similar magnetic field. The aim of this simulation is to see how particles behave in the Earth magnetic field. This could help to understand certain phenomena, like why the aurora is only happening around the poles.

2. Drift of charged particles in a magnetic field

The force acting on an electrical charged particle with mass m and charge q that moves in an electric field $\mathbf{E}$ and a magnetic field is $\mathbf{B}$ is given by the Lorentz force $F = q\,(\mathbf{E} + v \times \mathbf{B})$, which gives the equation of motion:

$$m\,\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1}$$

2.1. Solving linear ODEs numerically

Linear ODEs are equations on the form

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0 = 0 \tag{2}$$

where x^n is the n-th derivative of x:

$$x^n := \frac{d^n x(t)}{dt^n} \tag{3}$$

The Lorentz equation (1) can be written

$$\frac{\mathrm{d}^2\hat{\mathbf{x}}}{\mathrm{d}\hat{t}^2} = \mathbf{E} + \hat{\mathbf{v}} \times \hat{\mathbf{B}}, \tag{4}$$

where $\hat{\mathbf{x}} = \frac{\mathbf{x}}{r_L}$ and $\hat{t} = \omega_c\,t$. ω_c is called the *cyclotron frequency* and is given by $\omega_c = \frac{|q|B}{m}$. r_L is the Larmor radius given by $r_L = v_\perp/\omega_c$, where $v_\perp$ is the speed perpendicular to the magnetic field.

Equation (4) can be rewritten in a more compact form by defining a state vector $\mathbf{X}$. Equation (4) then becomes

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\hat{t}} = f(\mathbf{X}).$$

Set the state vector $\mathbf{X} = [\vec{x},\, \vec{v}]$. By taking the derivative with respect to t, $f(\mathbf{X})$ becomes

$$f(\mathbf{X}) = \begin{bmatrix} \mathbf{v} \\ \mathbf{E} + \hat{\mathbf{v}} \times \hat{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ E_x \\ E_y \\ E_z \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \\ v_z \\ v_y B_z - v_z B_y \\ v_z B_x - v_x B_z \\ v_x B_y - v_y B_x \end{bmatrix} \tag{5}$$

where $\mathbf{B} = \mathbf{B}(\mathbf{x},\, t)$ and $\mathbf{E} = \mathbf{E}(\mathbf{x},\, t)$ in general.

2.2. Helmholtz coils

Two Helmholtz coils are placed with center on the same axis, here the $(O, \hat{\mathbf{e}}_z)$ axis with center at $Z = -d$ and $z = d$, such that they are facing each other. In cylindrical coordinated the magnetic field produced by the coils can be expressed in the form $\mathbf{B} = B_r(r,\, z)\hat{\mathbf{e}}_r + B_z(r,\, z)\hat{\mathbf{e}}_z$. Using the Biot-Savart law, B_r and B_z can be expressed as

$$\hat{B}_r(\hat{r}, \hat{z}) = \frac{\left(1 + \hat{R}^2\right)^{3/2}}{4\pi\hat{R}} \int_0^{2\pi} \frac{(\hat{z} - 1)\,\cos\theta}{\left((\hat{r} - \hat{R}\cos\theta)^2 + \hat{R}^2\sin^2\theta + (\hat{z} - 1)^2\right)^{3/2}}\mathrm{d}\theta + \int_0^{2\pi} \frac{(\hat{z} + 1)\,\cos\theta}{\left((\hat{r} - \hat{R}\cos\theta)^2 + \hat{R}^2\sin^2\theta + (\hat{z} + 1)^2\right)^{3/2}}\mathrm{d}\theta$$

$$\hat{B}_z(\hat{r},\hat{z}) = \frac{\left(1+\hat{R}^2\right)^{3/2}}{4\pi} \int_0^{2\pi} \frac{1-\frac{\hat{r}}{\hat{R}}\cos\theta}{\left((\hat{r}-\hat{R}\cos\theta)^2+\hat{R}^2\sin^2\theta+(\hat{z}-1)^2\right)^{3/2}} d\theta + \int_0^{2\pi} \frac{1-\frac{\hat{r}}{\hat{R}}\cos\theta}{\left((\hat{r}-\hat{R}\cos\theta)^2+\hat{R}^2\sin^2\theta+(\hat{z}+1)^2\right)^{3/2}} d\theta$$

Such a setup is called a magnetic bottle and are able to trap a charged particle, *i.e.* a proton or an electron.

2.3. The Earth magnetic field

The Earth magnetic field can be modeled as a magnetic dipole and is best described in spherical coordinates. The magnetic field can be expressed in the form $\mathbf{B}(\mathbf{x}) = B_r(r,\theta)\hat{\mathbf{e}}_r + B_\theta(r,\theta)\hat{\mathbf{e}}_\theta$. B_r and B_θ is given by

$$B_r = -2B_0\cos\theta\frac{r_e^3}{r^3}, \tag{6}$$

$$B_\theta = -2\sin\theta\frac{r_e^3}{r^3}. \tag{7}$$

where $r = \sqrt{x^2+y^2+z^2}$, r_e is the Earth radius and B_0 is the magnetic field at equator. By introducing the reduced parameters: $\hat{r} = r/r_e$, $\hat{t} = \omega t$, $\hat{\mathbf{x}} = \mathbf{x}/r_e$, $\hat{\mathbf{v}} = \mathbf{v}/c_0$, $\hat{\mathbf{B}} = \mathbf{B}/B_0$, $\omega_{e,p} = eB_0/m_{e,p}$ and $\alpha_{e,p} = \omega_{e,p}/\omega$ the equation of motion becomes:

$$\frac{d^2\hat{\mathbf{x}}}{d\hat{t}^2} = \alpha_{e,p}\hat{\mathbf{v}}\times\hat{\mathbf{B}} \tag{8}$$

2.4. Magnetic moment

The magnetic moment to the particle along the field is given by $\mu = IS$, where I is the current caused by the charged particle moving in a loop during the time $\tau_c = 2\pi/\omega_c$ and $S = \pi r_L^2$ is the area of the loop. This gives the magnetic moment

$$\mu = -\frac{|q|}{\tau_c}\pi r_L^2 = -\mathrm{sgn}(q)\,\pi r_L^2\,\frac{q\omega_c}{2\pi} = -\mathrm{sgn}(q)\,\frac{qv_\perp^2}{3\omega_c} \tag{9}$$

3. Results & Discussion

The ODE was solved numerically using the Runge-Kutta 4 (RK4) method. The integrals in the magnetic field for the Helmholtz coils was solved using existing methods from the `SciPy` package [2].

3.1. Magnetic bottle

Figure 1 shows the trajectory of a proton trapped between two Helmholtz coils.

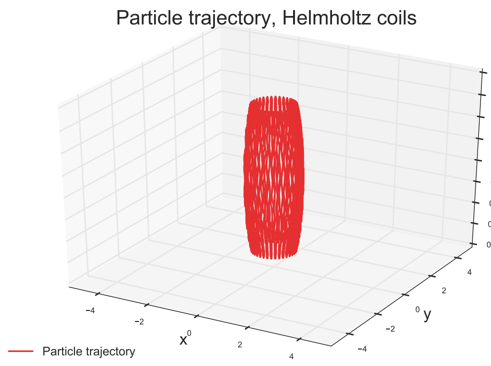

Figure 1: Proton trajectory between two Helmholtz coils. initial conditions: $\mathbf{x} = (1,0,1)$, $\mathbf{v}_0 = (1/\sqrt{2},0,1/\sqrt{2})$. Reduced units are used.

The magnetic moment for the first few seconds is shown in Figure 2.

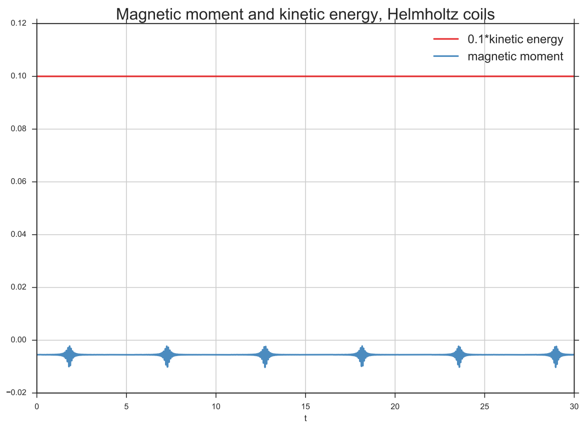

Figure 2: Magnetic moment and relative kinetic energy for the first few seconds. Proton between two Helmholtz coils.

From Fig 2 it seems that the magnetic moment is fairly conserved throughout the simulation. There also seems to be some sort of oscillation, but i don't know why this is happening. The kinetic energy seems conserved.

3.2. The Earth magnetic field

Figure 3 shows the trajectory of a proton moving in the Earth magnetic field.

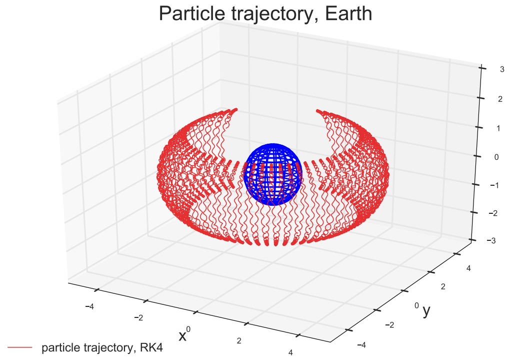

Figure 3: Proton trajectory moving in the Earth magnetic field. The Earth is visualised to scale by the blue sphere. Initial position $\vec{x}_0 = (0, 4r_e, 0)$ and initial velocity $\vec{v}_0 = \frac{0.08c_0}{\sqrt{2}}(1, 0, 1)$.

The magnetic moment for the first few seconds is shown in Figure 4.

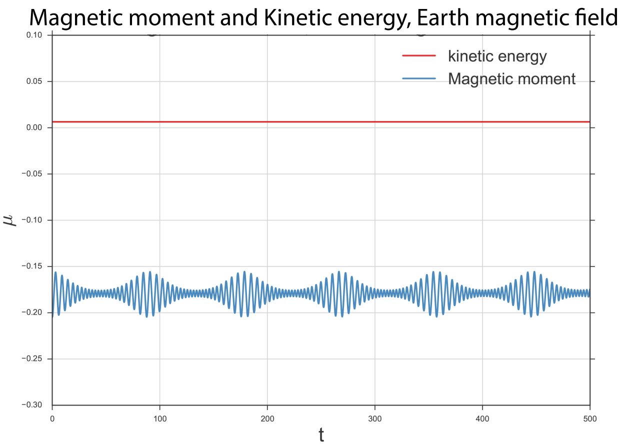

Figure 4: Magnetic moment and relative kinetic energy for the first few seconds. Proton in the Earth magnetic field.

Similar as for the Helmholtz coils, the kinetic energy is conserved. The magnetic moment also seems conserved, but have some small oscillations.

4. Conclusion

This simulation shows that when solar winds hit Earth with charged particles, the particles will 'bounce' from pole to pole and precess around the earth. The particles tends to be closer to the origin (the center of Earth) when they turn around at the poles, which explains why the Aurora is more likely to happen in a circle around the magnetic north and south pole.

References

[1] Problem Set 2, TFY4235, March 2016
[2] `http://www.scipy.org/`, v. 0.17

5. Appendix

The code was tested on several simpler electric and magnetic fields as shown in the figures below.

Figure 5 and 6 shows the particle trajectory of a charged particle in a constant electric and magnetic field calculated using Euler, midpoint and RK4 scheme.

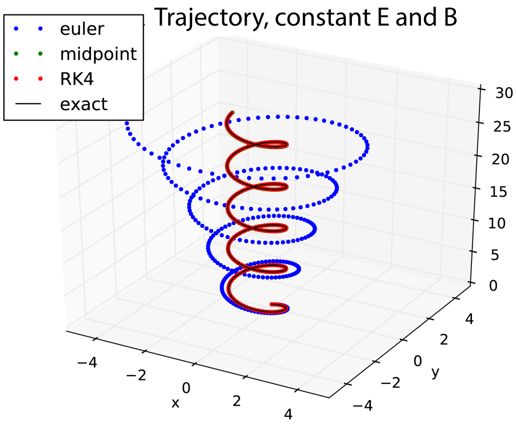

Figure 5

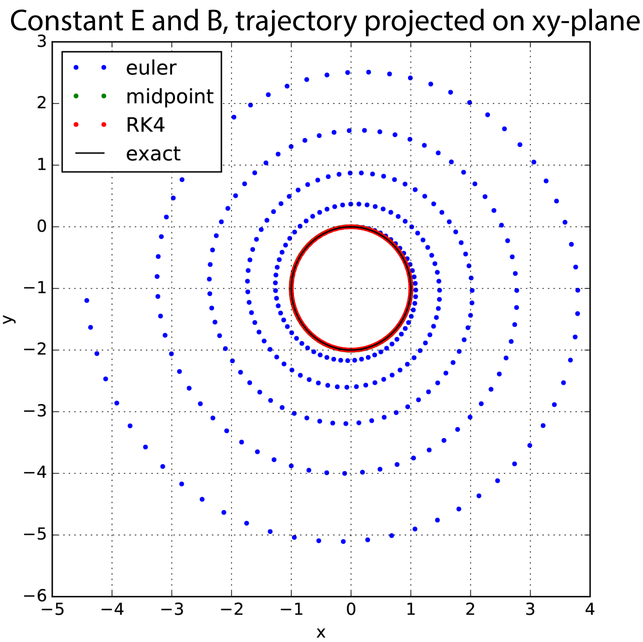

Figure 6

Figure 7 shows the numerical error as a function of step size. Note that for step size smaller than around 10e − 3 the RK4 scheme does not give a more accurate result due to the precision of the variables used.

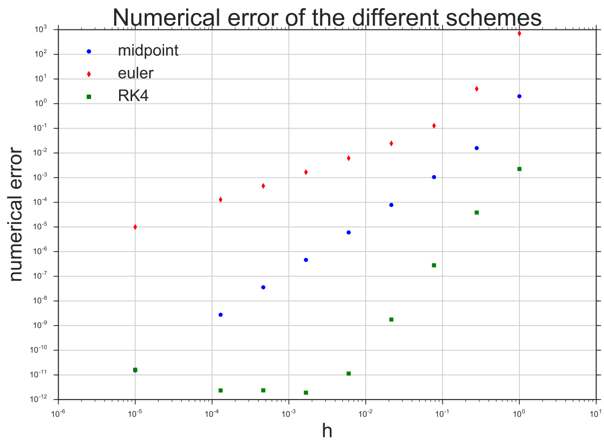

Figure 7: Numerical error using the different schemes.

Figure 8 shows the trajectory of a proton and an electron in a magnetic field with constant gradient linear in y; $B(y) = B_0 + \beta y$.

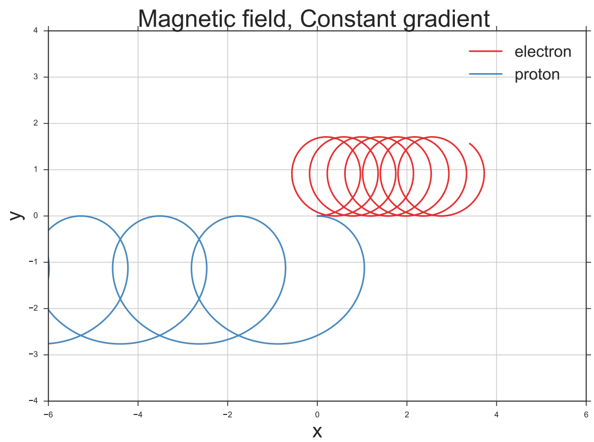

Figure 8: Magnetic field with constant gradient. Both particles has initial position at the origin with initial velocity (1,0,0). Using RK4 scheme.

Figure 9 shows a particle moving in a curved field expressed as $\mathbf{B} = B\hat{\mathbf{e}}_\theta$.

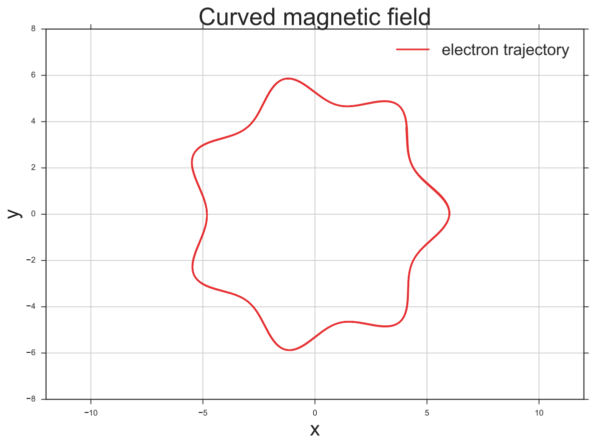

Figure 9: Particle in a curved magnetic field. Using RK4 scheme.

All the particle trajectories looks reasonable.